明新科技大學　校內專題研究計畫成果報告

廣義超立方體圖之最大引導子圖
Maximum induced subgraph of the generalized hypercube graphs

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廣義超立方體圖之最大引導子圖

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中文摘要

一個連結網路(interconnection network)的拓譜結構(topological structure)可以用一個圖型 $G=(V,E)$來表示，其中 V 代表圖型 G 的節點(vertex)集合、E 代表圖型 G 的鏈結(edge)集合。圖型 G 的節點子集合以 V'表示，我們定義 $G[V']$是在圖型 G 中節點 V'引導出的子圖(subgraph)，$G[V']$是包含節點 V'以及在節點 V'中任兩節點間所組成的所有鏈結(edge)圖型。一個圖型的 m-引導子圖是經由給定 m 個節點所引導出的圖形。一個圖型 G 的最大引導子圖我們以 $V_m^{\max}(G)$ 表示，其定義為 $V_m^{\max}(G)=\{\, G[V'] \mid \max_{V'\subseteq V,|V'|=m} |E(G[V'])| \,\}$。令 $\max_m(G)$是最大引導子圖 $V_m^{\max}(G)$ 中包含的鏈結數目。圖型 G 的最大引導子圖對於網路容錯(fault tolerance)與頻寬(bandwidth)之估算有其應用。令 m 是一個正整數，其中 $m=\sum_{i=0}^{r-1}2^{l_i}$ 且 $l_0>l_1>\cdots>l_{r-1}$，我們定義 $g(m)=\sum_{i=0}^{r-1}(\frac{l_i}{2}+i)2^{l_i}$。2003 年 Abdel-Ghaffar 證明了在 n-維超立方體(hypercubes)Q_n 中，當 $n\geq 1$ 且 $0\leq m\leq 2^n$ 時，$\max_m(Q_n)=g(m)$。在那之後，遞迴環狀圖(recursive circulant graphs)的最大引導子圖也在 2005 年由 X. Yang 等人提出。我們以遞迴的方式對 $n\geq 0$ 的廣義超立方體(generalized hypercubes) GQ_n 做一個定義。所有超立方體，雙扭立方體、交叉立方體與梅氏立方體都是廣義超立方體的特例。在本研究中，我們證明當 $n\geq 3$ 且 $0\leq m\leq 2^n$ 時，$\max_m(GQ_n)=g(m)$。我們也提出一個演算法可以找到廣義超立方體的最大引導子圖。

關 鍵 詞：最大引導子圖、超立方體、雙扭立方體、交叉立方體、梅氏立方體、廣義超立方體。

Maximum induced subgraph of the generalized hypercube graphs

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Abstract

The topological structure of an interconnection network can be modeled by a graph $G = (V, E)$ where V is the vertex set and E the edge set of G. For a vertex subset V' of graph G, the subgraph of G induced by V', denoted by $G[V']$, is a graph with vertex set V' and all the edges of G with both ends of vertices in V'. An *m-induced subgraph* of a graph is such one which induced by m vertices. A *maximum m-induced subgraph* of a graph G, denoted by $V_m^{\max}(G)$, can be defined as $V_m^{\max}(G) = \{ G[V'] \mid \max_{V' \subseteq V, |V'| = m} |E(G[V'])| \}$.

Let $\max_m(G)$ be the number of edges in such a maximum m-induced subgraph $V_m^{\max}(G)$.

Maximum m-induced subgraph of graph G has applications in the evaluation of *fault tolerance* and *bandwidth* of networks. Let m be an integer with $m = \sum_{i=0}^{r-1} 2^{l_i}$ and $l_0 > l_1 > \cdots > l_{r-1}$. $g(m) = \sum_{i=0}^{r-1} (\frac{l_i}{2} + i) 2^{l_i}$. For an n-dimensional *hypercube* Q_n, it is proved by Abdel-Ghaffar in 2003 that $\max_m(Q_n) = g(m)$ for $n \geq 1$ and $0 \leq m \leq 2^n$. After that, maximum m-induced subgraph of the *recursive circulant graphs* has proposed by X. Yang et al. in 2005. We recursively define *generalized hypercubes*, denoted by GQ_n, for $n \geq 0$. All of hypercubes, *twisted cubes*, *crossed cubes*, and *möbius cubes* are special cases of generalized hypercubes. In this paper, we prove that $\max_m(GQ_n) = g(m)$ for $n \geq 3$ and $0 \leq m \leq 2^n$. We also provide an algorithm to find the maximum m-induced subgraph of generalized hypercubes.

Keywords: maximum m-induced subgraph, hypercubes, twisted cubes, crossed cubes, möbius cubes, generalized hypercubes.

目　　　錄

1 Introduction

The topological structure of an *interconnection network* can be modeled by a *graph*, while vertices represent *processors* and edges represent *links* between processors. For the purpose of connecting hundreds or thousands of processing elements, many interconnection network topologies have been proposed in literature. Graph theory can be used to analyze the networks and most of the graph definitions we use are standard [4]. Terms networks and graphs are used interchangeably in this paper.

Given a graph $G = (V, E)$ where V is the vertex set and E the edge set of G. For a vertex subset V' of graph G, the subgraph of G induced by V', denoted by $G[V']$, is a graph with vertex set V' and all the edges of G with both ends of vertices in V'. An *m-induced subgraph* of a graph is such one which induced by m vertices [16]. A *maximally m-induced subgraph* of a graph G, denoted by $V_m^{\max}(G)$, can be defined as

$$V_m^{\max}(G) = \{G[V'] \mid \max_{V' \subseteq V, |V'|=m} |E(G[V'])|\}.$$

Let $\max_m(G)$ be the number of edges in such a maximally m-induced subgraph $V_m^{\max}(G)$. The *n-dimensional hypercube* [3], denoted by Q_n, is an undirected graph with 2^n vertices, which consists of all n-bit binary strings as its vertices. Take Q_3 for an instance, let V_1 be the vertex set $\{000, 001, 011, 111\}$, then $E(Q_3[V_1]) = \{(000, 001), (001, 011), (011, 111)\}$. However, let V_2 be the vertex set $\{000, 001, 011, 010\}$, then $E(Q_3[V_2]) = \{(000, 001), (001, 011), (011, 010), (010, 000)\}$. Actually, $\max_4(Q_3) = 4$. To maximize the number of edges joining vertices of a vertex set with m vertices of a graph is an important issue in this research.

Let $m = \sum_{i=0}^{r-1} 2^{l_i}$, where $l_0 > l_1 > \cdots > l_{r-1} \geq 0$. For an n-dimensional hypercube Q_n, it is proved in [1] that $\max_m(Q_n) = \sum_{i=0}^{r-1}(\frac{l_i}{2} + i)2^{l_i}$. For the *recursive circulant graphs* $RC(2^n, 4)$, the value of $\max_m(RC(2^n, 4))$ is proved in [16]. The results of maximally m-induced subgraph have applications in the evaluation of fault tolerance and bandwidth of networks. Maximizing the number of transitions corresponding to single edges may decrease the power consumption because of switching activities in processors [14]. In addition, they also relate to electromechanical or optical sensors [16]. The same technique also can be used to facilitate browsing of documents in libraries and data storage systems [12]. Some other applications can be seen in [1].

The n-dimensional hypercube, denoted by Q_n, is a popular network because of its attractive properties, including *regularity*, *symmetry*, *small diameter*, *strong connectivity*, *recursive construction*, *partitionability*, and *relatively low link complexity* [3, 11, 13]. There are some variations of the hypercube Q_n appearing in literature, such as *twisted cubes* [2, 10], *crossed cubes* [9, 15], and *möbius cubes* [8, 15]. These variations preserve most of the good topological properties of the hypercube, and even better. For example, the diameter of these variation cubes is around half of that of the hypercube. Recently, the twisted cubes, crossed cubes, and möbius cubes are proved to be super connected and super fault-tolerant hamiltonian graphs [5, 7]. We define a generalization of those graphs. The n-dimensional *generalized hypercubes*, denoted by GQ_n, are generalizations of the the n-dimensional twisted cubes TQ_n, crossed cubes CQ_n, and möbius cubes MQ_n. Let r and $l_0 > l_1 > \cdots > l_{r-1}$ be nonnegative integers with $m = \sum_{i=0}^{r-1} 2^{l_i}$. In this paper, we show that $\max_m(GQ_n) = \sum_{i=0}^{r-1}(\frac{l_i}{2}+i)2^{l_i}$ for $n \geq 3$ and $0 \leq m \leq 2^n$. Moreover, we provide an algorithm to find the maximally m-induced subgraph $V_m^{max}(GQ_n)$ of graph GQ_n for $n \geq 3$ and $0 \leq m \leq 2^n$.

The rest of this paper is organized as follows. Section 2 starts with the definition of generalized hypercubes and defines the function $g(m)$. Section 3 forms the main result of the paper. In Section 4, we give the conclusion remarks.

2 Preliminary

Motivated by the recursively structure of the hypercubes, crossed cubes, twisted cubes, and möbius cubes, we have the following n-dimensional generalized hypercubes, denoted by GQ_n. The GQ_n for $n \geq 0$ is recursively defined as follows. For $n = 0$, GQ_0 is a vertex. For $n = 1$, GQ_1 is isomorphic to the 1-dimensional hypercube Q_1 with vertex set $\{0, 1\}$ and edge set $\{(0, 1)\}$. As for $n \geq 2$, GQ_n consists of (1) two not necessarily identical GQ_{n-1}'s, denoted by GQ_{n-1}^0 and GQ_{n-1}^1; and (2) an arbitrary perfect matching with 2^{n-1} edges between the two GQ_{n-1}'s, each vertex in GQ_{n-1}^0 is adjacent to exactly one vertex in GQ_{n-1}^1. The n-dimensional generalized hypercubes GQ_n for $n = 1, 2$ are shown in Figure 1, in which the edge set of GQ_2 has two different situations. Figure 2 illustrates labels of the vertex set of GQ_n.

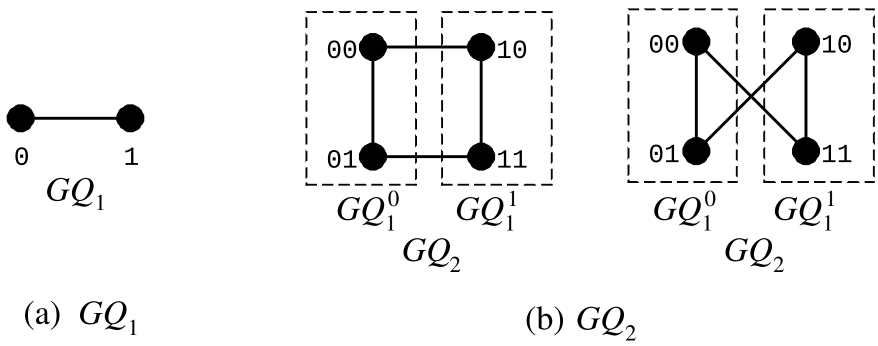

Figure 1: (a) GQ_1; (b) Two situations of GQ_2.

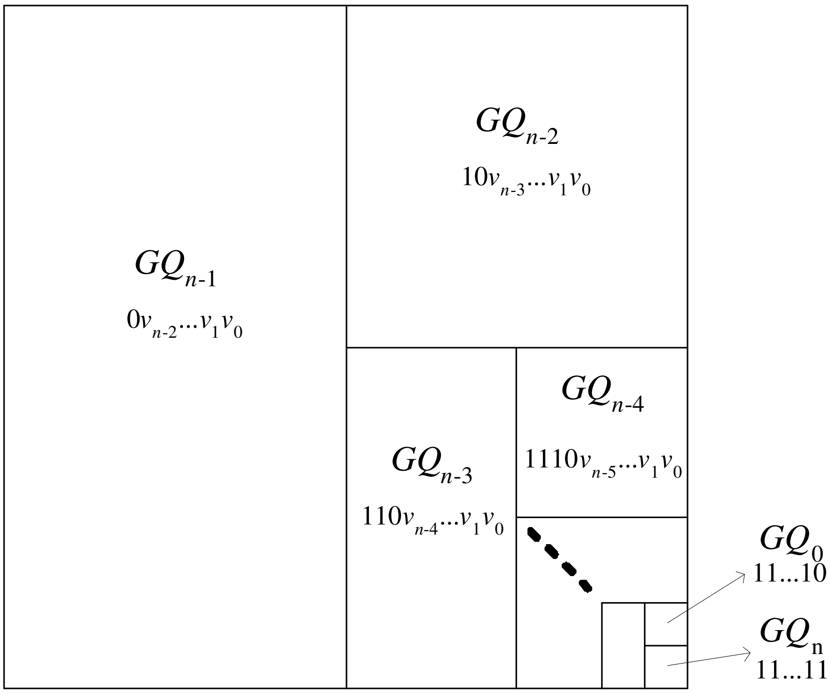

Figure 2: Labels of the vertex set of GQ_n.

Now, the Function $g(m)$ is defined as the following. Let m be an integer with $m = \sum_{i=0}^{r-1} 2^{l_i}$ and $l_0 > l_1 > \cdots > l_{r-1}$. Then, $g(m) = \sum_{i=0}^{r-1}(\frac{l_i}{2} + i)2^{l_i}$. As an example, for $n = 86 = 2^6 + 2^4 + 2^2 + 2^1$, $g(86) = (6/2 + 0)2^6 + (4/2 + 1)2^4 + (2/2 + 2)2^2 + (1/2 + 3)2^1 = 259$.

3 Maximally m-induced Subgraph of Generalized Hypercubes

In this section, we state and show the main result that given a generalized hypercube GQ_n for $n \geq 3$ and an integer m for $0 \leq m \leq 2^n$, we have that $\max_m(GQ_n) = g(m)$. In order to prove it, the following lemma is needed.

Lemma 1 *[16] For any nonnegative integers m_0,m_1, $g(m_0+m_1) \geq g(m_0)+g(m_1)+\min\{m_0, m_1\}$.*

The following lemma shows that for $n \geq 3$ and $0 \leq m \leq 2^n$, the maximally m-induced subgraph $V_m^{\max}(GQ_n)$ of GQ_n contains at most $g(m)$ edges by induction.

Lemma 2 *Given a generalized hypercube GQ_n for $n \geq 3$, and an integer m for $0 \leq m \leq 2^n$. We have that $\max_m(GQ_n) \leq g(m)$.*

Proof. This lemma is proved by induction. For the induction base $n = 3$, it is not hard to check that $\max_m(GQ_3) \leq g(m)$ for $0 \leq m \leq 8$ by brute force. Assume that $\max_m(GQ_n) \leq g(m)$ for $0 \leq m \leq 2^n$. Now, we shall show that for the m-induced subgraph of the GQ_{n+1}, $\max_m(GQ_{n+1}) \leq g(m)$ for $0 \leq m \leq 2^{n+1}$. In the m-induced subgraph with m vertices of GQ_{n+1}, we may assume that there are m_0 vertices in GQ_n^0 and m_1 in GQ_n^1 with $m = m_0 + m_1$. Without loss of generality, we may assume that $m_0 \geq m_1 \geq 0$. We divide the proof into the following two cases.

Case 1: $m_1 = 0$. So the m vertices are all distributed in GQ_n^0 and $m \leq 2^n$. By the induction hypothesis, we have that $\max_m(GQ_{n+1}) \leq g(m)$.

Case 2: $m_1 > 0$. For the maximally m-induced subgraph of GQ_{n+1}, there are $m_0 > 0$ vertices in GQ_n^0 and m_1 in GQ_n^1. Hence, $\max_m(GQ_{n+1}) \leq \max_{m_0}(GQ_n^0) + \max_{m_1}(GQ_n^1) + \min\{m_0, m_1\}$. By the induction hypothesis, $\max_{m_0}(GQ_n^0) \leq g(m_0)$ and $\max_{m_1}(GQ_n^1) \leq g(m_1)$. In addition, $g(m_0) + g(m_1) + \min\{m_0, m_1\} \leq g(m_0 + m_1)$ by Lemma 1. As a result, we have the following equation and this lemma is proved.

$$\begin{aligned}
\max_m(GQ_{n+1}) &\leq \max_{m_0}(GQ_n^0) + \max_{m_1}(GQ_n^1) \\
&\quad + \min\{m_0, m_1\} \\
&\leq g(m_0) + g(m_1) + \min\{m_0, m_1\} \\
&\leq g(m_0 + m_1) \\
&= g(m). \qquad \diamond
\end{aligned}$$

Now, we give an algorithm to find an m-induced subgraph of GQ_n with $g(m)$ edges.

Algorithm

01. $V' :=$ BUILD_VERTEX_SET(n, m);

/* Give an n-dimensional generalized GQ_n for $n \geq 3$, and an integer m, where $0 \leq m \leq 2^n$.

Let $m = \sum_{i=0}^{r-1} 2^{l_i}$, where $l_0 > l_1 > \cdots > 0 l_{r-1} \geq 0$. */

02. BUILD_VERTEX_SET(n, m)

03. begin

04. $\quad V' := \emptyset$; $\quad$ /* V' is a vertex subset of GQ_n. */

05. $\quad$ if $(m = 0)$ then return $V' := \emptyset$;

06. $\quad$ if $(m = 2^n)$ then return $V' := V(GQ_n)$;

07. $\quad$ for $i := 0$ to $r - 1$

08. $\quad\quad V' := V' \cup \{1^{n-(l_i+1)} 0 v_{l_i - 1} \cdots v_1 v_0 | v_x \in \{0, 1\}$ for $0 \leq x \leq l_i - 1\}$;

09. $\quad$ return V'; /* $GQ_n[V']$ is the maximally m-induced subgraph of GQ_n */

10. end BUILD_VERTEX_SET

Take one situation of GQ_3 as Figure 3 for example, while $m = 7$, $V' = \{000, 001, 010, 011, 100, 101, 110\}$. Now, we investigate in the number of edges of $GQ_n[V']$ of the above algorithm. Firstly, if $m = 0$, by Line 5 of the algorithm, $V' = \emptyset$ and $|E(GQ_n[V'])| = 0 = g(0)$. Secondly, if $m = 2^n$, by Line 6 of the algorithm, $V' = V(GQ_n)$ and $|E(GQ_n[V'])| = 2^n \times \frac{n}{2} = g(2^n)$. Finally, we consider that $0 < m < 2^n$. Let $m = \sum_{i=0}^{r-1} 2^{l_i}$, where $l_0 > l_1 > \cdots > l_{r-1} \geq 0$. After

Table 1: Total number of edges of $GQ_n[V']$ with $0 < m < 2^n$.

for loop	$\lvert E(GQ_n[V'])\rvert$
$i = 0$	$\lvert E(GQ_{l_0})\rvert$
$i = 1$	$\lvert E(GQ_{l_0})\rvert + (\lvert E(GQ_{l_1})\rvert + 2^{l_1})$
$i = 2$	$\lvert E(GQ_{l_0})\rvert + (\lvert E(GQ_{l_1})\rvert + 2^{l_1})$ $+ (\lvert E(GQ_{l_2})\rvert + 2 \times 2^{l_2})$
$i = 3$	$\lvert E(GQ_{l_0})\rvert + (\lvert E(GQ_{l_1})\rvert + 2^{l_1})$ $+ (\lvert E(GQ_{l_2})\rvert + 2 \times 2^{l_2})$ $+ (\lvert E(GQ_{l_3})\rvert + 3 \times 2^{l_3})$
$\cdots$	$\cdots$
$i = r - 1$	$\lvert E(GQ_{l_0})\rvert + (\lvert E(GQ_{l_1})\rvert + 2^{l_1})$ $+ (\lvert E(GQ_{l_2})\rvert + 2 \times 2^{l_2})$ $+ (\lvert E(GQ_{l_3})\rvert + 3 \times 2^{l_3})$ $+ \cdots$ $+ (\lvert E(GQ_{l_{r-1}})\rvert + (r-1) \times 2^{l_{r-1}})$ $= 2^{l_0} \times \frac{l_0}{2} + (2^{l_1} \times \frac{l_1}{2} + 2^{l_1})$ $+ (2^{l_2} \times \frac{l_2}{2} + 2 \times 2^{l_2})$ $+ (2^{l_3} \times \frac{l_3}{2} + 3 \times 2^{l_3})$ $+ \cdots$ $+ (2^{l_{r-1}} \times \frac{l_{r-1}}{2} + (r-1) \times 2^{l_{r-1}})$ $= g(m)$

finishing the for loop of the algorithm (lines 7-8), Table 1 is established, and the total number of edges of $GQ_n[V']$ is $g(m)$. Therefore, Lemma 3 follows.

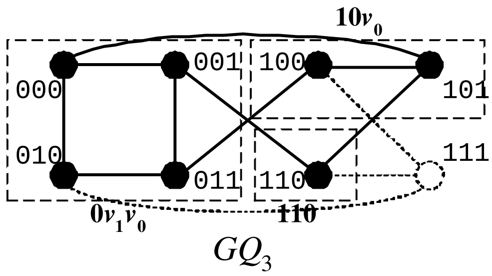

Figure 3: Maximally m-induced subgraph of the generalized hypercube GQ_3 with $m = 7$.

Lemma 3 *Given a generalized hypercube GQ_n for $n \geq 3$, and an integer m for $0 \leq m \leq 2^n$. We have that* $\max_m(GQ_n) \geq g(m)$.

According to Lemma 2 and Lemma 3, the main result of this paper is stated as Theorem 1.

Theorem 1 *Given a generalized hypercube* GQ_n *for* $n \geq 3$*, and an integer* m *for* $0 \leq m \leq 2^n$*. We have that* $\max_m(GQ_n) = g(m)$*.*

By the construction scheme of generalized hypercubes, the hypercubes, crossed cubes , twisted cubes, and möbius cubes are special cases of generalized hypercubes. As a result, we have the following corollary.

Corollary 1 $\max_m(Q_n) = \max_m(CQ_n) = \max_m(TQ_n) = \max_m(MQ_n) = g(m)$ *for* $n \geq 3$ *and* $0 \leq m \leq 2^n$*.*

4 Conclusion Remarks

The n-dimensional generalized hypercube GQ_n is a promising candidate for interconnection networks. Additionally, the crossed cubes CQ_n, twisted cubes TQ_n, and möbius cubes MQ_n are special cases of the GQ_n. This research determined the maximum number of edges of a subgraph of the GQ_n induced by a given number of m vertices with $0 \leq m \leq 2^n$. We also give an algorithm to find the maximally m-induced subgraph $V_m^{\max}(GQ_n)$ of generalized hypercubes.

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明新科技大學 97 年度 研究計畫執行成果自評表

計畫類別：□任務導向計畫 □整合型計畫 ☑個人計畫
所屬院(部)：□工學院 ☑管理學院 □服務學院 □通識教育部
執行系別：資管系
計畫主持人：陳玉專　　職稱：助理教授
計畫名稱：廣義超立方體圖之最大引導子圖
計畫編號：MUST-97-資管-01
計畫執行時間：97 年 1 月 1 日 至 97 年 9 月 30 日

計畫執行成效	教學方面	1. 對於改進教學成果方面之具體成效： 對於未來所授之相關課程，會有更加豐富的內容可與學生分享 2. 對於提昇學生論文/專題研究能力之具體成效： 參與的人員除了能夠得到實際的成果外，也能培養出濃厚的研究興趣及熱誠，並在研究的過程中，發掘更深更廣的研究方向，以作為未來研究的一個標地。 3. 其他方面之具體成效：
	學術研究方面	1. 該計畫是否有衍生出其他計畫案 □是 ☑否 計畫名稱： 2. 該計畫是否有產生論文並發表☑已發表 □預定投稿/審查中 □否 發表期刊(研討會)名稱：The 2008 International Computer Symposium 發表期刊(研討會)日期：97年 12 月18~20日 3. 該計畫是否有要衍生產學合作案、專利、技術移轉 □是 ☑否 請說明衍生項目：
成果自評		計畫預期目標：針對知名網路圖形，我們希望求得網路之最大引導子圖，並建立演算法以找出這些網路的節點集合 ，並且對其情況連通度作更深入之探討。 計畫執行結果：在本研究中，我們證明當 $n \geq 3$ 且 $0 \leq m \leq 2^n$ 時，$\max_m(GQ_n) = g(m)$。我們也提出一個演算法可以找到廣義超立方體的最大引導子圖。 預期目標達成率：90%
		其它具體成效： 網路之最大引導子圖在網路頻寬及容錯方面皆有其應用。另外，在一個網路的子圖中，若包含較多的鏈結數目，則對於處理器的資料交換行為也可達到節電的效果。未來，對於其他的知名網路圖形，可繼續延伸其最大引導子圖課題，以及探討其應用。